

The $(3, 1)$ -ordering for 4-connected planar triangulations

Therese Biedl ^{*} Martin Derka ^{*}

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Abstract

Canonical orderings of planar graphs have frequently been used in graph drawing and other graph algorithms. In this paper we introduce the notion of an (r, s) -canonical order, which unifies many of the existing variants of canonical orderings. We then show that $(3, 1)$ -canonical ordering for 4-connected triangulations always exist; to our knowledge this variant of canonical ordering was not previously known. We use it to give much simpler proofs of two previously known graph drawing results for 4-connected planar triangulations, namely, rectangular duals and rectangle-of-influence drawings.

1 Background

A canonical ordering of a planar graph is a way of building the graph by iteratively attaching vertices to some “basic graph” (such as an edge) while preserving some connectivity invariant after each iteration. This concept was introduced in the late 1980’s by de Fraysseix, Pach and Pollack [dFPP90]. They used the canonical ordering to show that planar graphs can be drawn on a grid of size $(2n - 4) \times (n - 2)$. Subsequently, canonical orderings became one of the main tools in graph drawings, e.g. for drawing graphs in grids of small dimensions (see e.g. [dFPP90, CN98]), rectangular duals [KH97], and also graph algorithms such as encoding planar graphs [HKL99] or finding k -disjoint trees in planar graphs [NRN97, NN00].

Our contribution There is now a number of variations of canonical orderings, depending on the connectivity of the graph and whether it is triangulated or not. (We will review these below.) In this paper, we show the existence yet another canonical ordering, this one for planar 4-connected triangulations. It is substantially different from the canonical ordering for such graphs that was presented by Kant and He [KH97]. We call this the $(3, 1)$ -canonical ordering. More generally, we introduce the concept of an (r, s) -canonical ordering, which (roughly speaking) means that the partial graph must be r -connected and the rest-graph must be s -connected; the existing canonical orders all fit into this framework.

^{*}David R. Cheriton School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada, {biedl, mderka}@uwaterloo.ca. Research supported by NSERC. The second author is supported by Vanier CGS.

We use the $(3, 1)$ -canonical ordering to provide alternate (and, in our opinion, significantly simpler) proofs of two previously known results about 4-connected planar triangulations: they have rectangular duals (Section 4.1) and rectangle-of-influence drawings (Section 4.2).

2 Review of existing canonical orderings

We assume that the reader is familiar with planar graphs (refer e.g. to [Die12]). We use the term *triangulation* for a maximal planar simple graph, i.e., a graph in which all faces are triangles and which has $3n - 6$ edges of which none is a multiple edge or a loop. Such a graph has a unique planar embedding; we further assume that one face has been fixed as the outer face. We begin our review of canonical ordering with the one for triangulations introduced by de Fraysseix et al. [dFPP90]. We paraphrase their definition to the following one (which is easily shown to be equivalent):

Definition 1 (Canonical ordering for triangulations [dFPP90]). *Let G be a triangulation with outer face u_1, u_2, u_3 . A vertex ordering v_1, \dots, v_n is called a canonical ordering if*

- $v_1 = u_1, v_2 = u_2, v_n = u_3$,
- For every $1 < k < n$ the subgraph G_k of G induced by vertices v_1, v_2, \dots, v_k is 2-connected.

As we will see later, it will be convenient to define $V_k := \{v_k\}$ and so $V_1 \cup \dots \cup V_n$ becomes a partition of the vertex set. For any such partition and an index k , we use the notation G_k for the subgraph induced by $V_1 \cup \dots \cup V_k$ and we let the *complement* $\overline{G_k}$ of G_k be the subgraph induced by the vertices $V - (V_1 \cup \dots \cup V_{k-1})$. Note that vertex set V_k belongs to both G_k and $\overline{G_k}$.¹

One can observe that in a canonical ordering for a triangulation, the complement $\overline{G_k}$ is a connected graph for all $k < n$. This holds because any vertex $v_k \neq u_1, u_2, u_3$ is not on the outer face and so there must exist some minimal $k' > k$ where v_k is not on the outer face of $G_{k'}$. Due to the triangular faces, v_k receives an edge to $v_{k'}$, and iterating the argument, hence has a path within $\overline{G_k}$ that leads to v_n .

We note here, without giving details, that this canonical ordering has been generalized to 3-connected planar graphs that are not necessarily triangulated [Kan96], and also to non-planar 3-connected graphs (see [Sch14] and the references therein).

In 1997, Kant and He [KH97] showed that one can define a different canonical ordering for 4-connected triangulations, and used it to construct visibility representations of 4-connected planar graphs. Its definition, slightly paraphrased, is as follows:

Definition 2 (Canonical ordering for 4-connected triangulations [KH97]). *Let G be a 4-connected triangulation with outer face u_1, u_2, u_3 . A vertex order v_1, \dots, v_n is called a canonical ordering for 4-connected triangulations if*

¹Some references instead define $\overline{G_k}$ to be the subgraph induced by $V - (V_1 \cup \dots \cup V_k)$. This complicates stating some of the conditions.

- $v_1 = u_1, v_2 = u_2, v_n = u_3,$
- For every $1 < k < n$, graphs G_k and $\overline{G_k}$ are 2-connected.

This canonical ordering was extended to a canonical ordering for all planar 4-connected graphs (not necessarily triangulated) by Nakano, Rahman and Nishizeki [NRN97]. Versions of a canonical order for 4-connected non-planar graphs are also known [CLY05].

Going one higher in connectivity, Nagai and Nakano [NN00] introduced a canonical ordering for 5-connected triangulations. Here, vertices are added in sets that are sometimes more than a singleton. We need a definition. Let G be a graph where all interior faces are triangles. A *fan* of G is a subset of vertices z_1, \dots, z_f that induces a path with $\deg(z_i) = 3$ for all $i = 1, \dots, f$. We will only apply this concept if all vertices in the fan belong to the outer face of G . Since interior faces are triangles, it follows that for all z_i the third neighbor (i.e., the one not on the outer face) is the same vertex. See also Figure 1(right).

Definition 3 (Canonical ordering for 5-connected triangulations [NN00]). *Let G be a 5-connected triangulation with outer face u_1, u_2, u_3 . A partition of the vertices $V = V_1 \cup \dots \cup V_L$ is called a canonical ordering for 5-connected triangulations if*

- $V_1 = \{u_1, u_2\},$
- V_2 consists of all neighbors of u_1 and u_2 ,
- $V_L = \{u_3\},$
- V_{L-1} consists of all neighbors of u_3 ,
- For $2 < k < L - 1$, vertex set V_k is either a single vertex or a fan,
- For every $2 < k < L$, graph G_k is 3-connected and graph $\overline{G_k}$ is 2-connected.

This canonical ordering was used to find 5 independent spanning trees in 5-connected triangulations [NN00]. To our knowledge, it has not been generalized to planar 5-connected (not necessarily triangulated) graphs, and not to non-planar 5-connected graphs either. Since no planar graph is 6-connected, no canonical orderings for higher connectivity can exist for planar graphs.

Note that the three canonical orderings listed here are very similar, with the essence being the connectivity that is required of the subgraphs and their complements. In light of this, we aim to unify the three definitions with the following:

Definition 4 ((r, s) -canonical ordering). *Let G be a triangulation with outer-face $\{u_1, u_2, u_3\}$. We say that a vertex partition $V_1 \cup \dots \cup V_L$ is an (r, s) -canonical ordering if*

- u_1 belongs to V_1 and u_3 belongs to V_L , and
- for every $1 < k < L$, graph G_k is r -connected and $\overline{G_k}$ is s -connected.

Note that this definition is deliberately vague on the exact form that the vertex sets V_k must have. In particular, nothing prevents us (yet) from setting $L = 1$ and $V_1 = V$, which satisfies all conditions. The existing canonical orderings restrict V_k to be a singleton or, for 5-connected triangulations, fans. Thus the above definition should be viewed as a class of definitions, to be refined further by stating restrictions on the vertex sets V_k .

Rephrasing the existing canonical orders in the above terms, the canonical order for triangulations becomes a $(2, 1)$ -canonical ordering with only singletons, the one for 4-connected triangulations becomes a $(2, 2)$ -canonical ordering with only singletons, and the one for 5-connected triangulations becomes a $(3, 2)$ -canonical ordering with only singletons or fans. The reader will notice that the sum of the two numbers equals the connectivity of the graph. Pushing this further, one may ask whether any $(r+s)$ -connected triangulation has an (r, s) -canonical ordering such that each V_k has some simple form. Note that we may assume that $r \geq s$, since a reversal of an (r, s) -canonical ordering gives an (s, r) -canonical ordering. We study here $(3, 1)$ -canonical ordering for 4-connected triangulations, under the restriction that each V_k is a singleton or a fan. To our knowledge no such ordering was known before.

3 $(3, 1)$ -canonical orderings

We have already given the broad idea of a $(3, 1)$ -canonical ordering earlier. We re-state it here and give the specific restrictions imposed on the vertex sets. See also Figure 1.



Figure 1: A singleton V_k and a fan V_k in a $(3, 1)$ -canonical ordering.

Definition 5. Let G be a 4-connected triangulation with outer-face $\{u_1, u_2, u_3\}$. A $(3, 1)$ -canonical order with singletons and fans is a partition $V = V_1 \cup \dots \cup V_L$ such that

- $V_1 = \{u_1, u_2, z\}$, where z is the third vertex of the interior face adjacent to (u_1, u_2) .
- $V_L = \{u_3\}$.
- For any $1 < k < L$, set V_k is either a singleton or a fan.
- For any $1 < k < L$, graph G_k is 3-connected and $\overline{G_k}$ is connected.

In what follows, we will omit the “with singletons and fans”, as we will not study any other version of $(3, 1)$ -canonical orderings. Our main goal is to show that every 4-connected triangulation has such a $(3, 1)$ -canonical ordering. The proof of this proceeds by induction,

and we state the crucial lemma for the induction step separately first. We need a few definitions.

A plane graph is called a *triangulated disk* if every interior face is a triangle and the outer-face is a simple cycle. A triangulated disk is called *internally 4-connected* if its outer-face has no chord, and every triangle is a face. Observe that a triangle is an internally 4-connected triangulated disk, and so is any 4-connected triangulation. Also observe that a subgraph of an internally 4-connected triangulated disk is again an internally 4-connected triangulated disk if and only if its outer-face is a simple cycle that has no chord.

Lemma 1. *Let G be an internally 4-connected triangulated disk with $n \geq 4$. Let (u_1, u_2) be an edge on the outer-face. Then there exists a vertex set V' such that*

- V' contains only outer-face vertices, and none of u_1, u_2 .
- $G - V'$ is an internally 4-connected triangulated disk.
- V' is a singleton or a fan.

Proof. ² Enumerate the outer face vertices of G as $u_1 = c_1, c_2, \dots, c_\ell = u_2$ in clockwise order. Define a *2-leg* to be a path $c_i - x - c_j$ where $i < j - 1$ and x is not on the outer-face. Vertex x is called a *2-leg-center*. We always have at least one 2-leg (namely, the one consisting of $u_1 = c_1, u_2 = c_\ell$ and their common neighbor at the interior face incident to (u_1, u_2) ; this vertex is interior since G has no chord and at least 4 vertices).

We say that a 2-leg-center x *dominates* a 2-leg-center y if vertex y is strictly inside the cycle $x - c_i - c_{i+1} - \dots - c_j - x$ formed by some 2-leg $\{c_i, x, c_j\}$ with center-vertex x . See also Figure 2(left). The dominance-relationship is acyclic since any 2-leg with center-vertex y must enclose strictly fewer faces than the 2-leg $\{c_i, x, c_j\}$. Therefore we must have some *minimal* 2-leg-centers, which are the ones that do not dominate any other 2-leg-center.

By definition for any 2-leg $\{c_i, x, c_j\}$, we have $j \geq i + 2$ and so there exists at least one vertex between c_i and c_j on the outer-face. We say that a 2-leg $\{c_i, x, c_j\}$ is *basic* if the vertices c_{i+1}, \dots, c_{j-1} all have degree 3, and *complex* otherwise. Note that if $\{c_i, x, c_j\}$ is basic, then c_{i+1}, \dots, c_{j-1} form a fan and their common neighbor is x .

Let x be a minimal 2-leg center. We have two cases:

- All 2-legs containing x are basic.

Let $i \geq 1$ be minimal and $j \leq \ell$ be maximal such that x is adjacent to c_i and c_j . See also Figure 2(middle). Since x is a 2-leg-center, we have $i < j - 1$. By case assumption the 2-leg $\{c_i, x, c_j\}$ is basic, so $V' = \{c_{i+1}, \dots, c_{j-1}\}$ is a fan. We verify that $G' := G - V'$ is an internally 4-connected triangulated disk:

- The outer-face of G' consists of the one of G plus x . By definition of a 2-center x was not on the outer-face, so G' is a triangulated disk.

²The proof is strongly inspired of the one for a $(3, 2)$ -canonical order in 5-connected graphs [NN00]. Since we demand less on our $(3, 1)$ -canonical order, we can simplify the exposition somewhat.

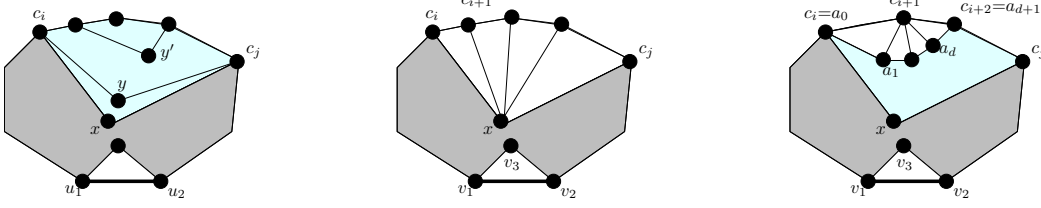


Figure 2: (Left) 2-leg center x dominates both y and y' . (Middle) If all 2-legs containing x are basic, then we can remove a fan. (Right) If $\{c_i, x, c_j\}$ is complex, then removing c_{i+1} leaves an internally 4-connected triangulated disk.

- Since G had no chord, the only possible chord of G' would be incident to vertex x . But by choice of i and j the only neighbors of x on the outer-face of G' are c_i and c_j . So G' has no chord.

- Some 2-leg $\{c_i, x, c_j\}$ is complex.

We assume that i has been chosen maximally, i.e., so that $\{c_{i+1}, x, c_j\}$ is either not a 2-leg or not complex. We claim that in this case $V' = \{c_{i+1}\}$ is a suitable vertex set.

We first show that c_{i+1} cannot be adjacent to x . Assume for contradiction that it is, then $\{c_i, x, c_{i+1}\}$ is a triangle and hence a face. If there were some c_h with $i+1 < h < j$ and $\deg(c_h) \geq 4$, then this would make $\{c_{i+1}, x, c_j\}$ a complex 2-leg, contradicting the choice of i . So all of c_{i+2}, \dots, c_{j-1} (if any) have degree 3, and they form a fan with common neighbor x . In particular, edge (c_{i+2}, x) exists, which means triangle $\{c_{i+1}, x, c_{i+2}\}$ is a face, forcing $\deg(c_{i+1}) = 3$. But then $\{c_i, x, c_j\}$ is basic, not complex. This is a contradiction, so x is not a neighbor of c_{i+1} .

Let $c_i = a_0, a_1, \dots, a_d, a_{d+1} = c_{i+2}$ be the neighbors of c_{i+1} in ccw order. See also Figure 2(right). None of a_1, \dots, a_d can be on the outer-face of G , else G would have a chord. The outer-face of $G' := G - V'$ consists of $c_1, \dots, c_i, a_1, \dots, a_d, c_{i+1}, \dots, c_\ell$, and so this is a simple cycle and G' is a triangulated disk. Further, we can show that it has no chord:

- If a chord of G' connected two vertices in $c_1, \dots, c_i, c_{i+2}, \dots, c_\ell$, then it would also be a chord in G , which is excluded.
- If a chord connected two non-consecutive vertices in $c_i = a_0, \dots, a_{d+1} = c_{i+2}$, then in G there would be an edge between two non-consecutive neighbors of c_{i+1} , implying a triangle that is not a face.
- If a chord connected some a_s , $1 \leq s \leq d$, with some c_h , $i+2 < h \leq j$, then $\{c_{i+1}, a_s, c_h\}$ would be a 2-leg in G . By minimality of x hence $a_s = x$, but this contradicts that c_{i+1} is not adjacent to x .
- If a chord connected some a_s , $1 \leq s \leq d$, with some c_h , $1 \leq h < i$ or $j < h \leq \ell$, then by $a_s \neq x$ it would have to cross (c_i, x) or (x, c_j) , contradicting planarity.

So G' is an internally 4-connected triangulated disk.

Observe that in both cases $V' \subseteq \{c_{i+1}, \dots, c_{j-1}\}$ for some $1 \leq i < j \leq \ell$, and so V' does not contain u_1 or u_2 as desired. \square

Theorem 1. *Let G be a 4-connected planar triangulation. Then G has a $(3, 1)$ -canonical order.*

Proof. We choose the vertex set in reverse order. Let $\{u_1, u_2, u_3\}$ be the outer-face and choose $V_L := \{u_3\}$; this satisfies all conditions since u_3 has at least 3 neighbors. (We do not at this point know the correct value of L , but simply assign indices backwards and shift indices at the end so that the vertex sets are numbered V_1, \dots, V_L .)

Observe that $G - u_3$ is an internally 4-connected triangulated disk, because the neighbors of u_3 form a simple cycle without chord (else there would be a separating triangle at u_3). Assume now some V_{k+1}, \dots, V_L have been chosen already such that the remaining graph $G_k := G - (V_{k+1} \cup \dots \cup V_L)$ is an internally 4-connected triangulated disk with (u_1, u_2) on the outer-face. If G_k has at least 4 vertices, then apply Lemma 1 to find the next V_k . Graph $G_k - V_k$ is again internally 4-connected, so we can continue choosing vertex sets until only 3 vertices, including u_1 and u_2 , are left. Since the graph is still internally 4-connected, these vertices must be a triangle, and hence a face of G . So setting V_1 to be the three vertices of this triangle gives the desired ordering.

To observe that the required connectivity holds, note that any internally 4-connected graph is 3-connected since it is a triangulated disk without a chord. To see that $\overline{G_k}$ is connected, it suffices to show that every vertex except u_3 has a neighbor in a later vertex set; the set of these edges then forms a spanning tree in $\overline{G_k}$. The argument for this is nearly the same as for $(2, 1)$ -orderings. Clearly each of u_1, u_2 are adjacent to u_3 . For any vertex $z \neq u_1, u_2, u_3$, vertex z is not on the outer face of G , and hence there must exist some minimal k' such that z is on the outer face of $G_{k'-1}$, but not on the outer face of $G_{k'}$. Since faces are triangles, this implies that z is adjacent to some vertex in $V_{k'}$. By the above hence $\overline{G_k}$ is connected for any $1 < k < L$. \square

The proofs of the above results are constructive and lead to polynomial time algorithms. With suitable data structures to keep track of 2-leg-centers, it is not hard to see that a $(3, 1)$ -canonical ordering can be found in linear time; we omit the details.

4 Applications

In this section, we demonstrate two uses for the $(3, 1)$ -canonical ordering in graph drawing. Both results proved here were known before, but in our opinion the $(3, 1)$ -canonical ordering significantly simplifies the proof of these results.

4.1 Rectangular duals

A *rectangular dual drawing* (or *RD-drawing* for short) of a planar graph G consists of a set of interior-disjoint rectangles assigned to the vertices of G in such a way that the union of

the rectangles forms a rectangle without holes, and the rectangles assigned to vertices v and w touch in a non-zero-length line segment if and only if (v, w) is an edge. The following theorem has been proved repeatedly:

Theorem 2 ([Ung53, Tho84, KH97]). *Let G be a 4-connected triangulation, and let e be an edge on the outer-face of G . Then $G - e$ has a rectangular dual.*

Previous proofs on this result usually used the $(2, 2)$ -canonical ordering (or some equivalent characterization, such as regular edge labellings). We give here a different proof using the $(3, 1)$ -canonical ordering.

Proof. Let the outer-face be $\{u_1, u_2, u_3\}$, chosen such that $e = (u_1, u_2)$. Find a $(3, 1)$ -canonical ordering $V_1 \cup \dots \cup V_L$ of G . We now build the rectangular-dual drawing of $G - e$ by drawing $G_k - e$ for $k = 1, \dots, L$. By construction, $e = (u_1, u_2)$ is an edge on the outer-face of G_k , and we can hence enumerate the outer-face of G_k as $c_1^k, \dots, c_{\ell_k}^k$ with $c_1^k = u_1$ and $c_{\ell_k}^k = u_2$. We maintain the invariant that in the RD-drawing of G_k , the rectangles of $c_1^k, \dots, c_{\ell_k}^k$ all attach at the top side of the bounding box, in this order.

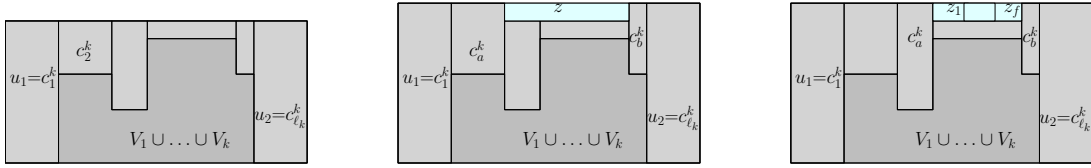


Figure 3: (Left) The invariant. (Middle and right) Adding a singleton and a fan.

Such a drawing is easily created for $G_1 - e$, since G_1 is a triangle and so $G_1 - e$ is a path $u_1 - z - u_2$, where z is the third vertex of the interior face at (u_1, u_2) . Now assume G_k is drawn and consider adding either a singleton or a fan V_{k+1} . Let a and b be the smallest and largest index such that c_a^k and c_b^k are adjacent to a vertex in V_{k+1} .

Extend all rectangles of c_1^k, \dots, c_a^k and $c_b^k, \dots, c_{\ell_k}^k$ upward by one unit. This leaves a “gap” where the rectangles of $c_{a+1}^k, \dots, c_{b-1}^k$ ended. There is at least one such rectangle since $b \geq a + 2$ by properties of the $(3, 1)$ -canonical ordering (else G_{k+1} would not be 3-connected). If V_{k+1} is a singleton z , then we insert the rectangle for z into this gap. If V_{k+1} is a fan $\{z_1, \dots, z_f\}$, then $b = a + 2$ and so the gap consists exactly of the top of c_{a+1}^k . Split this range into f pieces and assign rectangles for z_1, \dots, z_f in this place. One easily verifies that this represents all added edges as contacts and satisfies the invariant. So we have the desired RD-drawing. \square

4.2 Rectangle-of-influence drawings

A planar straight-line drawing of a graph is called a (*weak, closed*) *rectangle-of-influence drawing* (or *RI-drawing* for short) if for any edge (u, v) the *rectangle* $R(u, v)$ defined by u, v is *empty*, i.e., contains no other points of vertices of the graph. (It may contain parts of

other edges.) Here, $R(u, v)$ is the minimum axis-aligned rectangle that contains the points of u and v ; it degenerates into a line segment if u or v are on a horizontal or vertical line. The following result is known:

Theorem 3 ([BBM99]). *Let G be a 4-connected triangulation and let e be one edge of the outer-face. Then $G - e$ has a (weak, closed) rectangle-of-influence drawing.*

We re-prove this result using the $(3, 1)$ -canonical ordering. We note here that the drawing created is exactly the same as in [BBM99]; the difference lies in that we can find the next vertex set to add much more easily with the $(3, 1)$ -canonical ordering.

Proof. Let the outer-face be $\{u_1, u_2, u_3\}$, chosen such that $e = (u_1, u_2)$. Find a $(3, 1)$ -canonical ordering $V_1 \cup \dots \cup V_L$ of G . We now build the RI-drawing of $G - e$ by drawing $G_k - e$ for $k = 1, \dots, L$. By construction $e = (u_1, u_2)$ is an edge on the outer-face of G_k , and we can hence enumerate the outer-face of G_k as $c_1^k, \dots, c_{\ell_k}^k$ with $c_1^k = u_1$ and $c_{\ell_k}^k = u_2$. We maintain the invariant that in the RI-drawing of G_k

$$x(c_1^k) < x(c_2^k) < \dots < x(c_{\ell_k}^k) \quad \text{and} \quad y(c_1^k) > y(c_2^k) > \dots > y(c_{\ell_k}^k).$$

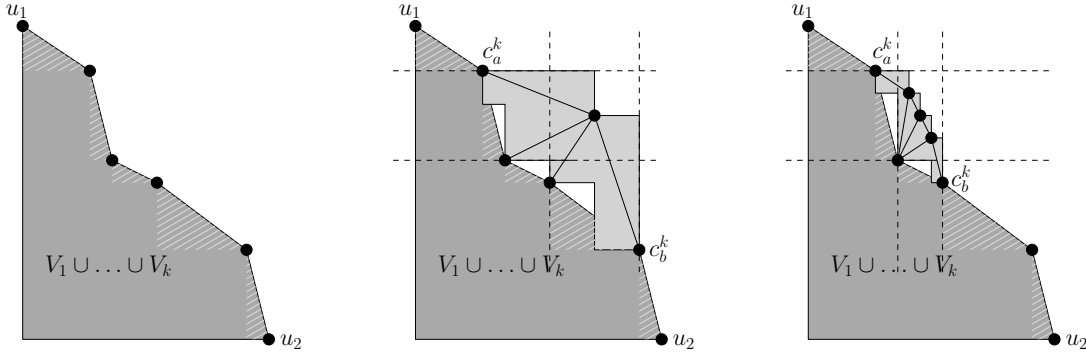


Figure 4: (Left) The invariant for RI-drawings. Hatched regions contain no points due to the RI-drawing. (Middle and right) Adding a singleton and a fan. The light gray region contains the new rectangles of influence.

Such a drawing is easily created for $G_1 - e$, since G_1 is a triangle and so $G_1 - e$ is a path $u_1 - z - u_2$, where z is the third vertex of the interior face at (u_1, u_2) . Now assume G_k is drawn and consider adding either a singleton or a fan V_{k+1} . Let a be the smallest and b be the largest index such that c_a^k and c_b^k are adjacent to a vertex in V_{k+1} . By 3-connectivity of G_{k+1} we have $b \geq a + 2$. If V_{k+1} is a singleton z , then define

$$x(z) = \frac{1}{2} (x(c_{b-1}^k) + x(c_b^k))$$

and

$$y(z) = \frac{1}{2} (y(c_a^k) + y(c_{a+1}^k)).$$

See also Figure 4(middle). By $a \leq b - 2$ adding this new point satisfies the invariant. All rectangles $R(z, c_j^k)$ are empty for $a \leq j \leq b$, because they do not intersect the drawing of G_k except in rectangles $R(c_a^k, c_{a+1}^k)$ and $R(c_{b-1}^k, c_b^k)$. So we have the desired RI-drawing.

If V_{k+1} is a fan $\{z_1, \dots, z_f\}$, then $b = a + 2$. For $h = 1, \dots, f$, define

$$x(z_h) = \frac{h}{f+1} (x(c_{b-1}^k) + x(c_b^k))$$

and

$$y(z_h) = \frac{f-h+1}{f+1} (y(c_a^k) + x(c_{a+1}^k)).$$

See also Figure 4(right). By $a = b - 2$ adding these new points satisfies the invariant. All rectangles $R(z_h, c_j^k)$ are empty for $a \leq j \leq b$, because they do not intersect the drawing of G_k except in rectangles $R(c_a^k, c_{a+1}^k)$ and $R(c_{b-1}^k, c_b^k)$. So we have the desired RI-drawing. \square

5 Conclusion

We showed the existence of new canonical order for 4-connected triangulations. We used this canonical order to give simplified proofs of some previously known graph drawing results for 4-connected triangulations. Furthermore, we provided a brief survey of canonical orderings for planar graphs and laid the groundwork for their further investigation. Of particular interest to us are the following questions:

- Does every planar c -connected triangulation have an (r, s) -canonical ordering for all $r + s = c$ and reasonable restrictions on vertex sets V_k ? The missing case is a $(4, 1)$ -canonical ordering for 5-connected triangulations.
- The (r, s) -canonical ordering definition naturally generalizes to planar graphs that are not necessarily triangulated. For the corresponding $(2, 1)$ -orderings [Kan96] and $(2, 2)$ -orderings [NRN97] it suffices to allow adding *chains*, i.e., induced paths. Are there $(3, 1)$ -orderings, $(3, 2)$ -orderings and $(4, 1)$ -orderings for 4-connected/5-connected planar graphs with some simple subgraphs as vertex sets V_k ? Likewise, exploration of (r, s) -canonical orders for non-planar graphs for $r + s \geq 5$ remains completely open.

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